A Group-Theoretical Approach to Disjoint Paths in Directed Graphs

Alexander Schrijver
CWI
Kruislaan 413, 1098 SJ Amsterdam
The Netherlands
email: lex@cwi.nl
and
Department of Mathematics, University of Amsterdam
Plantage Muidergracht 24, 1018 TV Amsterdam
The Netherlands.

We give a review of a some methods for finding disjoint paths in directed graphs. In particular, we describe a new group-theoretical method for finding disjoint paths in planar directed graphs.

1. Introduction

The *k disjoint paths problem* is as follows:

(1) given: a graph $D = (V, E)$ and k pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of D;
find: k pairwise vertex-disjoint paths $P_1, \ldots, P_k$ in D, where P_i runs from r_i to s_i $(i = 1, \ldots, k)$.

This is in general a hard problem. The problem is NP-complete (Knuth cf. [3]) even if we restrict ourselves to planar undirected graphs (Lynch [4]), assuming that k is variable. On the other hand, it is a deep result of Robertson and Seymour [6] that for each *fixed* k the problem can be solved in polynomial time in any undirected graph.

However, for *directed* graphs the problem remains difficult even for $k = 2$. This was shown by Fortune, Hopcroft and Wyllie [2]:

Theorem 1. *For directed graphs, the 2 disjoint paths problem is* NP*-complete.*

For certain subclasses of directed graphs, the k disjoint paths problem is solvable in polynomial time if we fix k. In 1980, Fortune, Hopcroft and Wyllie [2] showed that for fixed k the k disjoint paths problem can be solved for acyclic directed graphs (that is, directed graphs not having a directed cycle).

We describe their elegant argument in Section 2, together with an application to air plane routing.

Recently, we showed that the same holds for planar directed graphs (that is, directed graphs that can be embedded in the plane without intersecting arcs). The method is based on considering cohomology over combinatorial groups. We give a sketch of it in Section 3.

2. Acyclic graphs

We now show the result of Fortune, Hopcroft and Wyllie:

THEOREM 2. *For each k there exists a polynomial-time algorithm for the k disjoint paths problem for acyclic directed graphs.*

PROOF. Let $D = (V, A)$ be an acyclic digraph and let $r_1, s_1, \ldots, r_k, s_k$ be vertices of D, all distinct. In order to solve the disjoint paths problem we may assume that each r_i is a source and each s_i is a sink.

Make an auxiliary digraph $D' = (V', A')$ as follows. The vertex set V' consists of all k-tuples $(v_1, \ldots, v_k)$ of distinct vertices of D. In D' there is an arc from $(v_1, \ldots, v_k)$ to $(w_1, \ldots, w_k)$ if and only if there exists an $i \in \{1, \ldots, k\}$ such that:

(2) (i) $v_j = w_j$ for all $j \neq i$;
(ii) (v_i, w_i) is an arc of D;
(iii) if $j \neq i$ there is no directed path in D from v_j to v_i.

Now the following holds:

(3) D contains k disjoint directed paths $P_1, \ldots, P_k$ such that P_i runs from r_i to s_i $(i = 1, \ldots, k)$
$\Longleftrightarrow$ D' contains a directed path P from $(r_1, \ldots, r_k)$ to $(s_1, \ldots, s_k)$.

This implies the theorem, as a directed path in D' from $(r_1, \ldots, r_k)$ to $(s_1, \ldots, s_k)$ can be found in polynomial time if it exists. (Note that the size of D' might be large but still polynomially bounded in the size of D if k is fixed.)

To see $\Longrightarrow$, suppose that $P_1, \ldots, P_k$ exist. For any i, let P_i follow the vertices $v_{i,0}, v_{i,1}, \ldots, v_{i,t_i}$. So $v_{i,0} = r_i$ and $v_{i,t_i} = s_i$ for each i. Choose $j_1, \ldots, j_k$ such that $0 \leq j_i \leq t_i$ for each i and such that:

(4) (i) D' contains a directed path from $(r_1, \ldots, r_k)$ to $(v_{1,j_1}, \ldots, v_{k,j_k})$,
(ii) $j_1 + \cdots + j_k$ is as large as possible.

Let $I := \{i | j_i < t_i\}$. If $I = \emptyset$ we are done, so assume $I \neq \emptyset$. Then by the definition of D' and the maximality of $j_1 + \cdots + j_k$ there exists for each $i \in I$ an $i' \neq i$ such that there is a directed path in D from $v_{i',j_{i'}}$ to v_{i,j_i}. Since $s_{i'}$ is a sink we know that $v_{i',j_{i'}} \neq s_{i'}$ and that hence i' belongs to I. So each point in $\{v_{i,j_i} | i \in I\}$ is end point of a directed path in D starting in another point in $\{v_{i,j_i} | i \in I\}$. This contradicts the fact that D is acyclic.

To see $\Longleftarrow$ in (3), let P be a directed path from $(r_1, \ldots, r_k)$ to $(s_1, \ldots, s_k)$ in D'. Let P follow the vertices $(v_{1,j}, \ldots, v_{k,j})$ for $j = 0, \ldots, t$. So $v_{i,0} = r_i$ and $v_{i,t} = s_i$ for $i = 1, \ldots, k$. For each $i = 1, \ldots, k$ let P_i be the path in D following $v_{i,j}$ for $j = 0, \ldots, t$, taking repeated vertices only once. So P_i is a directed path from r_i to s_i.

Moreover, $P_1, \ldots, P_k$ are pairwise disjoint. For suppose that P_1 and P_2 (say) have a vertex in common. That is $v_{1,j} = v_{2,j'}$ for some $j \neq j'$. Without loss of generality, $j < j'$ and $v_{1,j} \neq v_{1,j+1}$. By definition of D', there is no directed path in D from $v_{2,j}$ to $v_{1,j}$. This however contradicts the facts that $v_{1,j} = v_{2,j'}$ and that there exists a directed path in D from $v_{2,j}$ to $v_{2,j'}$. □

One can derive from this that for fixed k also the k arc-disjoint paths problem is solvable in polynomial time for acyclic directed graphs (two paths are *arc-disjoint* if they do not have any arc in common):

COROLLARY 2A. *For each k there exists a polynomial-time algorithm for the k arc-disjoint paths problem for acyclic directed graphs.*

PROOF. Let $D = (V, A)$ be an acyclic directed graph and let $r_1, s_1, \ldots, r_k, s_k$. We reduce the k arc-disjoint paths problem to the k vertex-disjoint paths problem.

We may assume that for each $i = 1, \ldots, k$ there is exactly one arc, a_i say, leaving r_i and exactly one arc, b_i say, entering s_i. We now make an auxiliary directed graph D' with vertex set A, where there is an arc from $a \in A$ to $a' \in A$ if the head of a is equal to the tail of a'.

Note that D' is acyclic again. Moreover, finding arc-disjoint paths as required in D is equivalent to finding vertex-disjoint paths $Q_1, \ldots, Q_k$ in D' where Q_i runs from a_i to b_i $(i = 1, \ldots, k)$. □

We describe one practical application of this technique. An airline company carries out a certain number of flights according to some fixed timetable, in a weekly cycle. The timetable is basically given by a flight number i (e.g. 562), a departure city dc_i (e.g. Vancouver), a departure time dt_i (e.g. Monday 23.15h), an arrival city ac_i (e.g. Tokyo), and an arrival time at_i (e.g. Tuesday 7.20h). All times include boarding and disembarking and preparing the plane for a next flight. Thus a plane with arrival time Tuesday 7.20h at city c, can be used for any flight from c with departure time from Tuesday 7.20h on.

The flights are carried out by n air planes of one type, which planes are denoted by $q_1, \ldots, q_n$. At each weekday there should be an air plane for maintenance at the home basis, from 6.00h till 18.00h. Legal rules prescribe which of the air planes $q_1, \ldots, q_n$ should be for maintenance at the home basis during which day the coming week (depending on the number of flight hours).

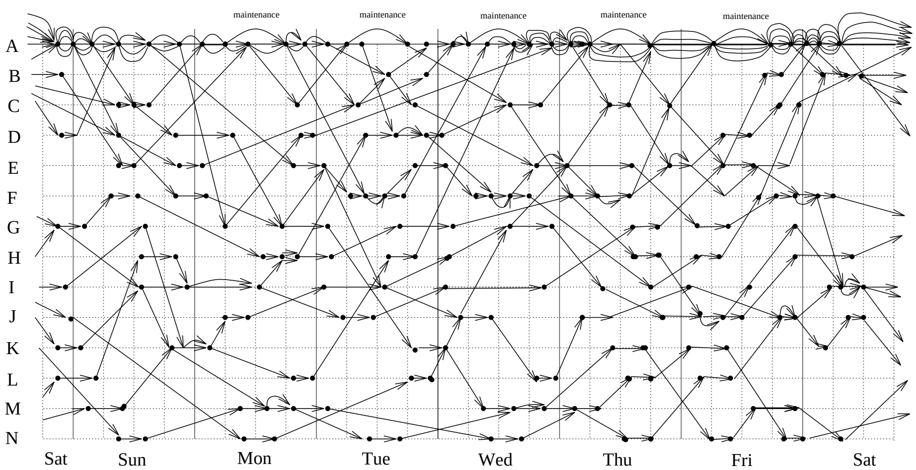
The timetable is made in such a way that at each city the number of incoming flights is equal to the number of outgoing flights. Here 'maintenance' is also considered as a flight.

However, there is flexibility in assigning the air planes to the flights: if at a certain moment at a certain city two or more air planes are available for a flight, in principle any of them can be used for that flight. Which of the available air planes will be used, is decided by the main office of the company. This decision is made on the Saturday before at 18.00h. At that time the company makes the exact routing of the planes for the coming week.

At that moment certain planes are performing certain flights, while other planes are grounded at certain cities. If there were no maintenance constraints, routing the air planes is easy, since the timetable is set up in such a way that at each moment and each city enough air planes are available.

Indeed, one can make a directed graph D with vertex set all pairs (dc_i, dt_i) and (ac_i, at_i) for all flight numbers i. For each flight i that is not in the air at Saturday 18.00h, one makes an arc from (dc_i, dt_i) to (ac_i, at_i). This is also done for the "flights" representing maintenance.

Moreover, for each city c and each two consecutive times t, t' at which any flight departs or arrives at c, one makes m parallel arcs from (c, t) to (c, t'), where m is the number of air planes that will be in city c during the period t–t'.



In this way we obtain a directed graph with the property that at each vertex the indegree is equal to the outdegree, except at any (c, t_c) where t_c is the earliest time after Saturday 18.00h at which any flight arrives at or leaves city c. Hence we can find in D arc-disjoint paths $P_1, \ldots, P_n$ (where n is the number of air planes) in D such that each arc is in exactly one of the P_i. This would give a routing for the air planes.

However, the restriction that some prescribed air planes must undergo maintenance the coming week gives some complications. It means e.g. that if a certain air plane q_i (say) is now on the ground at city c and should be home for maintenance the coming week, then path P_i should start at (c, t_c) and should traverse the arc representing maintenance of plane q_i. If q_i is now in the air, say flying to city c with arrival time t, then path P_i should start at (c, t) and should traverse the maintenance arc of plane q_i.

It follows that it is enough to find arc-disjoint paths $P_{i_1}, \ldots, P_{i_k}$ for those air planes $q_{i_1}, \ldots, q_{i_k}$ that should undergo maintenance the coming week. By the indegree = outdegree property, these paths can be extended to paths $P_1, \ldots, P_n$ such that each arc is traversed exactly once.

So the problem can be solved by finding arc-disjoint paths starting in a given set of vertices and ending in a given set of vertices. As the directed graph D is acyclic, these paths can be determined with the algorithm described in the proofs above, provided that the number of air planes that should be home for maintenance the coming week is not too large. (Note that it is not necessary to construct the whole graph D' in advance; the necessary part of it can be made during execution of the algorithm.)

3. Planar graphs

A second class of directed graphs where for each fixed k the disjoint paths problem is solvable in polynomial time is that of the *planar* graphs (Schrijver [7]).

The method is based on the following auxiliary framework concerning cohomology on graphs.

Let $D = (V, A)$ be a directed graph and let G be a group. Two functions $\phi, \psi : A \longrightarrow G$ are called *cohomologous* if there exists a function $p : V \longrightarrow G$ such that

$$(5) \quad \psi(a) = p(u)^{-1}\phi(a)p(w)$$

for each arc $a = (u, w)$. One directly checks that this gives an equivalence relation.

Consider the following *cohomology feasibility problem*:

(6) given: a directed graph $D = (V, A)$, a group G, a function $\phi : A \longrightarrow G$, and a subset H of G;
find: a function $\psi : A \longrightarrow H$ cohomologous to ϕ.

In general this is a difficult problem, even when G is the group $\{1, a, a^2\}$ with three elements, $H = \{a, a^2\}$ and $\phi(a) = 1$ for each arc a. In that case, the existence of ψ is equivalent to the 3-vertex-colorability of the graph D, which is an NP-complete problem.

However, there are cases where the cohomology feasibility problem can be shown to be solvable in polynomial time. We here consider the case where G is a *free group*, that is, a group generated by a finite set of generators $g_1, \ldots, g_k$, where G consists of all 'words' w with symbols from $g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}$, so that w does not contain any subword $g_i g_i^{-1}$ or $g_i^{-1} g_i$. (Word w' is a *subword* of w if $w = uw'v$ for some words u, v.)

Multiplication is defined by concatenation of words and successive cancellation of any subword $g_i g_i^{-1}$ or $g_i^{-1} g_i$. Then G is a group, with unit 1 equal to the empty word.

We call a subset H of a free group G an *ideal* if $1 \in H$ and with any word w in H also any subword of w belongs to H.

The following was proved in [7]:

THEOREM 3. *The cohomology feasibility problem is solvable in polynomial time if G is a free group and H is an ideal.*

(In fact, a more general result can be shown. We may replace 'free group' by 'free partially commutative group'. This is a group generated by generators $g_1, \ldots, g_k$ where for some pairs g_i, g_j of generators the relation

(7) $\quad g_i g_j = g_j g_i$

holds. Moreover, one may assume that for each arc a of D an ideal $H(a)$ is given such that $\psi(a) \in H(a)$ for each arc a is required in the cohomology feasibility problem. In our formulation above, $H(a)$ is independent of a.)

The proof of Theorem 3 consists of reducing it to the 2-satisfiability problem. Conversely, the 2-satisfiability problem can be seen as a special case of the cohomology feasibility problem for free groups. To see this, first note that any instance of the 2-satisfiability problem can be described as one of solving a system of inequalities in $\{0,1\}$ variables $x_1, \ldots, x_n$ of the form:

(8) $\quad x_i + x_j \geq 1$ for each $\{i,j\} \in E$,
$\quad\quad x_i + x_j \leq 1$ for each $\{i,j\} \in E'$,

where E and E' are given collections of pairs from $\{1, \ldots, n\}$.

Let G be the free group generated by the elements g and h. Make a directed graph with vertices $v_1, \ldots, v_n$ and with arcs:

(9) (i) $a = (v_i, v_j)$, with $\phi(a) := ghg^{-1}$, for each $\{i,j\} \in E$;

(ii) $a = (v_i, v_j)$, with $\phi(a) := h$, for each $\{i,j\} \in E'$.

Moreover, set $H = \{w \in G | \, |w| \leq 2\}$. (Here $|w|$ is the word length of w.)

Now the cohomology feasibility problem in this case is equivalent to solving (8) in $\{0,1\}$ variables. Indeed, if $x_1, \ldots, x_n$ is a solution of (8) then define $p(v_i) := g$ if $x_i = 1$ and $p(v_i) := 1$ if $x_i = 0$. Then ψ defined by (5) has values in H only.

Conversely, let $\psi : A \longrightarrow H$ and $p : V \longrightarrow G$ satisfy (5) Define $x_i := 1$ if $p(v_i) \neq 1$ and the first symbol of $p(v_i)$ is equal to g, and $x_i := 0$ otherwise. Then $x_1, \ldots, x_n$ is a solution of (8).

We will not go further into the details of the proof of Theorem 3, but show how the theorem implies the following theorem:

THEOREM 4. *For each k there exists a polynomial-time algorithm for the k disjoint paths problem for planar directed graphs.*

SKETCH OF PROOF. I. We first observe that the following is a consequence of Theorem 3. Let G be the free group with generators $g_1, \ldots, g_k$ and let

(10) $\quad H := \{1, g_1, \ldots, g_k\}$ and $I := \{1, g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}\}$.

Let $D = (V, A)$ be a directed graph, let $\phi : A \longrightarrow G$ and let $B \subseteq A$. Then we can find in polynomial time a function $\psi : A \longrightarrow I$ such that ψ is cohomologous to ϕ and such that $\psi(a) \in H$ for each $a \in B$.

To see this, make for each arc $a = (u, w)$ in $A \setminus B$ two new vertices v', v'', and replace a by four new arcs $a' = (u, v'), a'' = (w, v'), a''' = (v'', u)$ and $a'''' = (v'', w)$.

Let $D' = (V', A')$ be the graph arising this way. Let $\phi'(a) := \phi(a)$ for each unmodified arc a. For the new arcs let $\phi'(a') := \phi'(a'''') := \phi(a)$ and $\phi'(a'') := \phi'(a''') := 1$. One directly shows that there exists a function ψ as required, if and only if there exists a function $\psi' : A' \longrightarrow H$ cohomologous to ϕ'. (This follows from the fact that $I = HH^{-1} \cap H^{-1}H$.) This gives the reduction to the cohomology feasibility problem, since H is an ideal.

II. Let input $D = (V, A), r_1, s_1, \ldots, r_k, s_k \in V$ for (1) be given. We may assume that D is weakly connected, and that $r_1, s_1, \ldots, r_k, s_k$ are distinct, each being incident with exactly one arc. Fix an embedding of D in the plane, and let $\mathcal{F}$ denote the collection of faces of D.

Let G be the free group with k generators $g_1, \ldots, g_k$. Call two function $\phi, \psi : A \longrightarrow G$ *homologous* if there exists a function $f : \mathcal{F} \longrightarrow G$ such that

(11) $\quad f(F)^{-1}\phi(a)f(F') = \psi(a)$

for each arc a, where F and F' are the faces at the left-hand side and at the right-hand side of a, respectively (with respect to the orientation of a).

For any solution $\Pi = (P_1, \ldots, P_k)$ of (1) let $\phi_\Pi : A \longrightarrow G$ be defined by:

(12) $\quad \phi_\Pi(a) := g_i$ if path P_i traverses a $(i = 1, \ldots, k)$, and
$\phi_\Pi(a) := 1$ if a is not traversed by any of the P_i.

Now one can show:

(13) For each fixed k we can find in polynomial time functions $\phi_1, \ldots, \phi_N : A \longrightarrow G$ with the property that for each solution Π of (1), ϕ_Π is homologous to at least one of $\phi_1, \ldots, \phi_N$.

Note that this statement does not imply that there exists a solution Π of (1).

It follows that it suffices to describe a polynomial-time method for the following problem:

(14) given: a function $\phi : A \longrightarrow G$;
find: a solution Π of (1) such that ϕ_Π is homologous to ϕ.

Indeed, we can apply such an algorithm to each ϕ_j in (13). If we do not find Π for any ϕ_j, then (1) has no solution.

In order to solve (14) with the cohomology feasibility algorithm, we consider the *dual* graph $D^* = (\mathcal{F}, A^*)$ of D, having as vertex set the collection $\mathcal{F}$ of faces of D, while for any arc a of D there is an arc of D^*, denoted by a^*, from the face of D at the left-hand side of a to the face at the right-hand side of a. Define for any function ϕ on A the function ϕ^* on A^* by

(15) $\quad \phi^*(a^*) := \phi(a)$

for each arc a of D. Then any two functions ϕ and ψ are homologous (in D), if and only if ϕ^* and ψ^* are cohomologous (in D^*).

We extend the dual graph D^* to the 'extended' dual graph $D^+ = (\mathcal{F}, A^+)$ by adding in each face of D^* all chords. (So D^+ need not be planar.) To be more precise, for any (undirected) path π on the boundary of any face of D^*, extend D^* with an arc from F to F', where F and F' are the beginning and end of π. We denote this arc by a_π. For any $\phi : A \longrightarrow G$ define $\phi^+ : A^+ \longrightarrow G$ by:

(16) $\phi^+(a^*) := \phi^*(a^*)$ for each arc a of D;
$\phi^+(a_\pi) := \phi^*(a_1)^{\varepsilon_1}\phi^*(a_2)^{\varepsilon_2}\cdots\phi^*(a_t)^{\varepsilon_t}$ for any path π as above, where $a_1, a_2, \ldots, a_t$ are the arcs traversed by π (in that order) and where $\varepsilon_i = +1$ if a_i is traversed by π in forward orientation and $\varepsilon_i = -1$ if a_i is traversed by π in backward orientation.

Now let input ϕ of problem (14) be given. By the observation made in part I above, we can find in polynomial time a function $\psi : A^+ \longrightarrow I$ that is cohomologous to ϕ^+ in D^+, with $\psi(a^*) \in H$ for each arc a of D, provided that such a ψ exists.

If we find such a function ψ, let P_i be any directed $r_i - s_i$ path traversing only arcs a satisfying $\psi(a^*) = g_i$ $(i = 1, \ldots, k)$. If such paths exist, they form a solution of (1): This implies that if a and b are arcs of D incident with a vertex v, then a^* and b^* are incident with one face of D^* and if $\psi(a^*) = g_i^{\pm 1}$ and $\psi(b^*) = g_j^{\pm 1}$, then $i = j$ (since for each path π as above we have $\phi^+(a_\pi) \in I$). Hence at most one of $P_1, \ldots, P_k$ traverses v.

If we do not find such a function ψ and such paths we may conclude that problem (14) has no solution. For suppose that ϕ_Π is homotopic to ϕ for some solution $\Pi = (P_1, \ldots, P_k)$ of (1). Then there exists a ψ as above, viz. $\psi := (\phi_\Pi)^+$. Moreover, for any ψ' cohomologous to $(\phi_\Pi)^+$ there exists for each $i = 1, \ldots, k$ a directed $r_i - s_i$ path P_i' traversing only arcs a such that g_i occurs in $\psi'(a^*)$. So we would find a solution, contradicting our assumption. $\square$

The method can be extended to give more general results. For instance, also the polynomial-time solvability for fixed k of the following problem can be derived:

(17) given: a directed planar graph $D = (V, A)$, k pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of D, and subsets $A_1, \ldots, A_k$ of A;

find: k pairwise disjoint directed paths $P_1, \ldots, P_k$ in D, where P_i runs from r_i to s_i and uses only arcs in A_i $(i = 1, \ldots, k)$.

More generally, consider the problem (cf. [1]):

(18) given: a directed planar graph $D = (V, A)$, k pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of D, subsets $A_1, \ldots, A_k$ of A, and a set E consisting of some pairs $\{i, j\}$ from $1, \ldots, k$;
find: k directed paths $P_1, \ldots, P_k$ in D, where P_i runs from r_i to s_i and uses only arcs in A_i $(i = 1, \ldots, k)$ and where P_i and P_j are disjoint if $\{i, j\} \in E$.

Also this problem is solvable in polynomial time for each fixed k. (Here we need the free partially commutative groups mentioned above.)

A special case of this is the following *disjoint trees problem*:

(19) given: a directed planar graph $D = (V, A)$ and k pairs $(r_1, S_1), \ldots, (r_k, S_k)$ with $r_1, \ldots, r_k \in V$ and $S_1, \ldots, S_k \subseteq V$;
find: k pairwise disjoint rooted trees $T_1, \ldots, T_k$ in D, where T_i has root r_i and covers S_i $(i = 1, \ldots, k)$.

This problem is solvable in polynomial time for any fixed upper bound K on the cardinality of $\{r_1, \ldots, r_k\} \cup S_1 \cup \cdots \cup S_k$.

Actually, in these results, it is not necessary to fix the number of terminals. It suffices to fix the minimum number t for which there exist t faces of the planar graph such that each of the terminals is on the boundary of at least one of these faces. (For the (undirected) case $t = 2$ RIPPHAUSEN, WAGNER and WEIHE [5] gave a *linear*-time algorithm.)

Moreover, the results hold for graphs embedded on any fixed surface. But it is unkwown if the 2 *arc*-disjoint paths problem can be solved in polynomial time for directed planar graphs.

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